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RENORMALIZATION-GROUP ANALYSIS OF TURBULENCE

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ABSTRACT

The renormalization-group (RG) analysis of turbulence, based primarily on KG Wilson's coarse-graining procedure, leads to suggestive results for turbulence coefficients and models. Application of the method to turbulence evolved from the contributions of many authors and received widespread attention following the 1986 work of V Yakhot and SA Orszag. The Yakhot-Orszag method involves the basic renormalization-group scale-removal procedure, as well as additional hypotheses and approximations; their analysis is reviewed here with an attempt to clarify those approximations. Discussion of some related and subsequent literature is also included. Following the work of M Avellaneda and AJ Majda, a simpler version of the method is appplied to a model passive scalar problem wherein it is seen that, in certain cases, the RG method can recover exact results.

1. INTRODUCTION

1.1 *Objective*

The application of renormalization-group (RG) techniques to turbulence has attracted a great deal of interest, which it deserves, but perhaps not enough

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understanding, which it also deserves. Extreme reactions to the RG technique run the gamut—from unthinking acceptance as a magical black box to complete rejection as errant nonsense. But the RG method is neither nonsense nor magic: It is simply an extremely powerful tool for physics calculations when applied to problems for which it is suited. The difficulty is that we cannot yet say with certainty that turbulence is one of those problems.

Given this situation, the authors have sought in this review to convey an understanding of the basic RG procedure and to explore as fully as possible the approximations required to apply it to turbulence. Additionally, it is hoped that placing the RG analysis of turbulence in the context of the method's origins in other areas of physics will give a clearer picture of its capabilities and limitations. Our fundamental thesis is that the RG method, a calculation procedure successfully employed in many areas of physics, leads to suggestive results when applied to turbulence and may well lead to important results in the future. However, its application to turbulence cannot yet be called a major success, owing to the uncontrolled approximations currently required to implement it.

A glance at the following pages will quickly convince the reader that this contribution is more mathematical than most. While we have endeavored to keep the mathematics as simple as possible, it must be recognized that the RG procedure is a technique of mathematical physics, like Fourier transforms and perturbation theory, and not an area of fluid mechanics in particular or physics in general. If the mathematics seem off-putting, please bear with us, for the basic principles of the technique are fairly simple and worth understanding, even if the details of the calculations remain opaque.

1.2 *Historical Background*

There was a period in the late 1920s when the problems of quantum phenomena that had vexed physicists for so long fell rapidly before the seemingly invincible new quantum mechanics of Heisenberg (1926) and Schrödinger (1926). But the beginning of the end for these heady days was signaled in the early 1930s when several perturbative calculations (e.g. Weisskopf 1934) in the theory of interactions between electrons and electromagnetic fields (Dirac 1928) yielded divergent integrals. This fundamental difficulty in the theory of quantum electrodynamics remained unresolved until just after World War II, when Tomonaga (1946), Schwinger (1948), and Feynman (1949) independently introduced the concept of renormalization: The divergent integrals could be absorbed into physical constants such as the mass and the charge of the electron. By assuming the finiteness of these new renormalized constants (replacing the original, "bare," values), predictions that agree with experiments to as many as 10 significant figures can be made.

To carry out this "dippy" procedure, as Feynman (1985) characterized it, the divergent integrals had to be made artificially finite by, for example, replacing

an integration limit of infinity by a constant μ. The RG technique was born when Stueckelberg & Petermann (1953) and Gell-Mann & Low (1954) showed that the fact that no observable physical quantity can be influenced by such an artifice could be used to extend the domain of validity of perturbative results. The derivative of an observable with respect to the artificial parameter μ must be zero; forcing the perturbation series to satisfy this condition has the effect of summing certain infinite subsets of terms, thus increasing the series' domain of validity. This is an elegant formulation of the RG procedure that is intimately bound up with the mathematical machinery of quantum field theories such as quantum electrodynamics, and therefore is beyond the scope of this review.

The study of critical phenomena spawned the other major form of the RG method. The computation of phase transitions in large systems is made difficult by the strong correlations between many molecules when the system is near transition. Kadanoff (1966) proposed the conceptual tool of reducing the size of the system a step at a time by grouping neighboring molecules and treating each group as a single molecule. Repetition of this process provides an intuitive basis for the scaling laws (analogous to the Kolmogorov theory of the inertial range) that play a large role in the description of critical phenomena. While this effective-molecule approach has been employed as a computational tool, the most successful implementation of the general idea of coarse-graining or "weeding out of small scales" has been that of Wilson (1970).

Wilson chose to work in Fourier space, to permit the easy sorting of large and small scales. By seeking only information about large-scale motion, the problem of many interacting scales becomes manageable if one can eliminate the large-wavenumber, small-scale features of the problem in favor of the small-wavenumber, large-scale features. Once all but the smallest wavenumbers are eliminated, one is left with a problem involving only large scales, wherein the effects of the smaller scales are represented by corrected, or renormalized, coefficients. Wilson showed how the small scales could be eliminated by dividing wavenumber space into shells—like an onion with its layers—and eliminating scales shell by shell beginning with the outermost. Each shell produces its own correction to the renormalized coefficients, and the final renormalized coefficients are the sums of the corrections from all the shells.

The successive elimination of the large number of thin shells necessary to leave only the large scales is accomplished by iteratively removing what is, at any given stage in the process, the outermost shell. The self-similar or power-law nature of the problems typically addressed with this technique manifests itself in that the elimination of one shell is identical to the elimination of any of the others, once the dependent and independent variables are rescaled to accommodate the change in shell size.

The errors incurred in the elimination of shells are controlled by introducing a perturbative approximation, now known simply as the ε expansion (e.g. Wilson

& Kogut 1974, Wilson 1975). Often couched as a perturbative expansion about a particular number of spatial dimensions or about a particular power-law exponent, this ε expansion is ultimately an approximation based on the smallness of the effect of long-range statistical fluctuations.

The mode-elimination RG method has been developed and modified by Wilson and others, and relationships with the quantum-field–theory version of the RG procedure have been established (Wilson & Kogut 1974). Of particular interest here is what is known as the dynamic RG technique (e.g. Ma & Mazenko 1975, Halperin et al 1974), which is concerned with applying mode-elimination ideas to evolution equations (such as the Navier-Stokes equations) instead of to the statistical-mechanical partition functions employed in the study of critical phenomena. (For comparisons of turbulence and critical phenomena, see Nelkin 1974, Rose & Sulem 1978, and Eyink & Goldenfeld 1994.)

With this background, the RG technique is a natural candidate for application to the problem of turbulence. The RG technique, as developed in the traditional physics literature, is primarily intended for determining scaling laws, so its role in turbulence might be expected to be the determination of quantities such as the exponent of the inertial-range energy spectrum. However, the Kolmogorov $-5/3$ exponent for the inertial-range spectrum is fixed by Kolmogorov's hypotheses (1941) and dimensional considerations, therefore any dimensionally correct calculation procedure that respects those hypotheses must give this exponent (Kraichnan 1982). Consequently, if the RG technique together with the Kolmogorov hypotheses is to make nontrivial predictions, it must predict coefficients such as the Kolmogorov constant.

Forster et al (1977) applied mode-elimination RG techniques to the Navier-Stokes equations stirred by random body forces to model, for example, a fluid near thermal equilibrium. DeDominicis & Martin (1979) employed the quantum-field–theory RG method to show that a random force with a power-law correlation leads to the Kolmogorov $-5/3$ energy spectrum for one choice of the power-law exponent. Then Fournier & Frisch (1983), by dispensing with the rescaling part of the basic mode-elimination procedure, were able to relate the Kolmogorov constant to the energy input rate of the random force. Yakhot & Orszag (1986) made these results useful in practice by relating the energy input rate of the random force to the dissipation rate of the turbulence, and so computed a value for Kolmogorov's constant in agreement with experimental values.

As a final historical observation, it is important to recognize that the RG method is far from the first application of the renormalization concept to turbulence. The direct interaction approximation (DIA) of Kraichnan (1959) has been interpreted in a variety of ways but ultimately may be considered an example of a renormalized perturbation theory (RPT). An RPT is, like the quantum-field–theory RG, an attempt to improve the behavior of a perturbative

solution by summing certain infinite subsets of terms. In the case of RPT, this is achieved by formulating the problem in renormalized dependent variables, such as a renormalized velocity correlation function. Further developments in the application of RPTs to turbulence include, for example, the diagrammatic expansions of Wyld (1961) and Lee (1965), the Lagrangian formulation for the DIA of Kraichnan (1965), and the recent RPT of L'vov & Procaccia (1995) involving a difference of velocities. General formulations of RPT for random classical systems (e.g. Phythian 1969, Martin et al 1973) have been proposed, providing a framework for both RPT and RG applications to turbulence. Because diagrams are heavily used in the development of both RPT and RG formulations, we also use diagrams herein to describe the RG analysis. Furthermore, we hope that the diagrammatic presentation of the RG technique will aid in understanding the relationship between the RG and RPT methods (Section 2.8).

1.3 *Overview*

Section 2 concerns primarily the Yakhot-Orszag (1986) application of the RG method to turbulence. The basic concepts of the calculation are discussed in some detail, and the emphasis is on understanding the nature of the approximations invoked and on the extent of their validity. In particular, we present a proper interpretation of the random force and its relationship to the other approximations.

One of the things that makes understanding the RG method difficult is the very thing that makes it useful: the RG method is employed to study behavior inherent in complex statistical systems. Thus, a problem that is simple to understand may be too simple to generate nontrivial behavior. Section 3 details the application of mode-elimination RG to a passive-scalar model problem of Avellaneda & Majda (1990), which does a fair job of avoiding this difficulty. They have also provided an exact renormalization theory (a rigorously correct solution valid for long times and large distances), which may be compared with RG results to test the RG technique's effectiveness.

Section 4 deals with developments in turbulence modeling that have resulted from RG analyses of turbulence.

We conclude with a brief summary.

We regret that the goals outlined above limit the scope of this article to the extent that the contributions of many authors cannot be included. Other discussions of RG as applied to turbulence are found in Lesieur (1990) and Frisch (1995), and a large part of McComb (1990) is devoted to the subject. Before proceeding, a small point on notation: It is customary in RG literature to work with dimensional equations, and we have chosen to follow this practice to ease the reader's transition to that literature. It will be seen that a nondimensional parameter nevertheless plays a crucial role in understanding the basis for the method.

2. RANDOMLY FORCED NAVIER-STOKES EQUATIONS

2.1 *Homogeneous Isotropic Turbulence*

We now present both the ideas and the actual formalism of the RG method as it has been applied to the problem of turbulence. The Navier-Stokes equations for incompressible flow are

$$\frac{\partial u_i}{\partial x_i} = 0; \quad \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \nu_o \frac{\partial^2 u_i}{\partial x_j \partial x_j}, \tag{1}$$

where ν_o is the kinematic viscosity, the constant density has been absorbed into the pressure p, and repeated indices indicate summation. These equations are characterized by one nondimensional parameter, the Reynolds number, measuring the relative strengths of inertial and viscous effects. The Reynolds number may be defined as $Re \equiv U_o L_o/\nu_o$, where U_o, L_o, and L_o^2/ν_o are, respectively, characteristic velocity, length, and time scales of the flow. At high Reynolds numbers, the nonlinearity of the Navier-Stokes equations tends to excite a large range of length and time scales, leading to turbulence. In fact, the ratio of largest to smallest length scales may be estimated to increase as $Re^{3/4}$ (Tennekes & Lumley 1972). Experiments indicate that the net energy transfer in three spatial dimensions is from large to small scales. At high Re, the cascade of energy from large to small scales takes place through a self-similar range of scales, called the inertial range, where viscous effects are negligible. The energy spectrum of the inertial range is observed to scale approximately as $E(k) \propto k^{-5/3}$ (Grant et al 1962). Because an analytical solution of turbulence subject to realistic initial and boundary conditions is beyond present capabilities, much research has focused on what many believe to be the simplest turbulence problem, that of statistically homogeneous and isotropic turbulence. The material of this section is concerned solely with this problem.

To discuss this idealized case of homogeneous isotropic turbulence, averages over independent realizations of the velocity field are introduced. A primary focus of analytical theories is the two-point correlation function $R_{ij}(\mathbf{x}, t, \mathbf{x}', t') = \langle u_i(\mathbf{x}, t)\ u_j(\mathbf{x}', t')\rangle$, where the angle brackets indicate the ensemble average. Homogeneity (isotropy) means that all averaged quantities are independent of a translation (rotation and reflection) of the coordinate axes. If, furthermore, statistical stationarity is assumed, ensemble averages depend only on time differences. Therefore the correlation function R_{ij} has the form $R_{ij} = R_{ij}(\mathbf{x} - \mathbf{x}', |t - t'|)$ (Orszag 1973).

The Fourier transform of the velocity field $\hat{u}_i(\hat{\mathbf{k}})$ is defined as

$$\hat{u}_i(\hat{\mathbf{k}}) = \int d\mathbf{x} \int dt\ u_i(\mathbf{x}, t) \exp(-i(\mathbf{k} \cdot \mathbf{x} - \omega t)), \tag{2}$$

where $\hat{\mathbf{k}}$ is the four-vector $(\mathbf{k}, \omega)$, and the integrals are over all time and all space (Lesieur 1990, McComb 1990). Then the energy spectrum of velocity fluctuations is defined as $E(k) = \Phi_{ii}(k)/2$; the energy-spectrum tensor $\Phi_{ij}(k)$ is the integral over angles in spherical coordinates of $\Phi_{ij}(\mathbf{k})$, the Fourier transform of the equal-time correlation $R_{ij}(\mathbf{x} - \mathbf{x}', 0)$.

The Navier-Stokes equations (Equation 1) may be written in terms of the Fourier modes, $\hat{u}_i(\hat{\mathbf{k}})$, as

$$\hat{u}_i(\hat{\mathbf{k}}) = \frac{-i}{2} G^{(0)}(\hat{\mathbf{k}}) P_{imn}(\mathbf{k}) \int \frac{d\hat{\mathbf{q}} d\hat{\mathbf{p}}}{(2\pi)^{d+1}} \hat{u}_m(\hat{\mathbf{q}}) \hat{u}_n(\hat{\mathbf{p}}) \delta(\hat{\mathbf{k}} - \hat{\mathbf{p}} - \hat{\mathbf{q}}), \tag{3}$$

where $G^{(0)}(\hat{\mathbf{k}})$, the Green's function or *bare propagator*, is given by

$$G^{(0)}(\hat{\mathbf{k}}) \equiv (-i\omega + \nu_o k^2)^{-1}. \tag{4}$$

Note that $G^{(0)}(\hat{\mathbf{k}})$ involves the bare, or molecular, viscosity. As for the other notation in Equation 3, d is the number of space dimensions, $P_{imn}(\mathbf{k}) \equiv k_m P_{in}(\mathbf{k}) + k_n P_{im}(\mathbf{k})$, $P_{ij}(\mathbf{k}) \equiv \delta_{ij} - k_i k_j / k^2$, $k = |\mathbf{k}|$, and the integration is over the entire $\hat{\mathbf{q}} = (\mathbf{q}, \Omega)$ domain. Equation 3 may be derived by first transforming Equation 1; the transformed continuity equation may then be used to eliminate the pressure from the divergence of the transformed Navier-Stokes equations (Orszag 1973). The convolution operator $\Gamma^{(0)}_{imn}(\hat{\mathbf{k}}, \hat{\mathbf{p}}, \hat{\mathbf{q}})$ of the nonlinear term in Equation 3 is called the *bare vertex*, given by

$$\Gamma^{(0)}_{imn}(\hat{\mathbf{k}}, \hat{\mathbf{p}}, \hat{\mathbf{q}}) = \frac{-i}{2} P_{imn}(\mathbf{k}) \int \frac{d\hat{\mathbf{p}} d\hat{\mathbf{q}}}{(2\pi)^{d+1}} \delta(\hat{\mathbf{k}} - \hat{\mathbf{p}} - \hat{\mathbf{q}}) \tag{5}$$

and acting upon the two velocities $\hat{u}_m(\hat{\mathbf{q}})$ and $\hat{u}_n(\hat{\mathbf{p}})$. Modified, or renormalized, propagators $G(\hat{\mathbf{k}})$ and renormalized vertices $\Gamma_{imn}(\hat{\mathbf{k}}, \hat{\mathbf{p}}, \hat{\mathbf{q}})$ are basic ingredients in RG (and RPT) theories of turbulence.

Equation 3 describes the decay of an isotropic, homogeneous, incompressible velocity field, because the use of Fourier transforms precludes the addition of energy through the large-scale boundary conditions. Often, however, sustained turbulence is of interest, so a body force is added to the momentum equation to provide the energy input necessary to maintain a statistically stationary state. This force is traditionally taken to be Gaussian and white-in-time; thus it is completely determined by its two-point correlation

$$\langle \hat{f}_i(\hat{\mathbf{k}}) \hat{f}_j(\hat{\mathbf{k}}') \rangle = 2D(k)(2\pi)^{d+1} P_{ij}(\mathbf{k}) \delta(\hat{\mathbf{k}} + \hat{\mathbf{k}}'). \tag{6}$$

The factor $\delta(\hat{\mathbf{k}} + \hat{\mathbf{k}}')$ ensures homogeneity in space and time and $P_{ij}(\mathbf{k})$ is the most general second-rank, isotropic, divergence-free tensor. [Because the velocity field is divergence-free, it can only be affected by the divergence-free part of the force (Orszag 1973).] The interpretation of the force defined by

Equation 6 depends largely on the form of the spectrum $D(k)$. For example, in three spatial dimensions the characteristics of the turbulent cascade of energy to small scales are of primary interest, and a large-scale stirring force may be employed, localized about some small wavenumber. Large-scale stirring sets up the inertial-range cascade, presumably independent of the details of the force when the Reynolds number is large.

A power-law spectrum $D(k)$ is typically used in RG calculations to establish the self-similarity necessary to use the method. As a model of a fluid near equilibrium, Forster et al (1977) considered $D(k) = D_o k^2$ for $k < \Lambda$ (their model A), where Λ is a cutoff wavenumber and D_o is a dimensional constant. In this case, the force simulates the effects of molecular randomness. They also considered $D(k) = D_o$ for $k < \Lambda$ (their model B), corresponding to what the authors call a macroscopic shaking of the fluid container. Extending this analysis, DeDominicis & Martin (1979) used $D(k) \propto k^{-y}$, and Fournier & Frisch (1983) took $D(k) = D_o k^{1-\varepsilon}$ for $\varepsilon \to 0$ in three dimensions. Yakhot & Orszag (1986) considered $D(k) = D_o k^{4-d-\varepsilon}$ for arbitrary dimension d, but focused on $d = 3$. For $\varepsilon > 0$, dimensional arguments give the energy spectrum $E(k) \propto k^{1-2\varepsilon/3}$ (Fournier & Frisch 1978, 1983, Yakhot & Orszag 1986, Frisch 1990); the value $\varepsilon = 4$ for $d = 3$ is implied by Kolmogorov's (1941) hypotheses and corresponds to the energy spectrum $E(k) \propto k^{-5/3}$. The important role of the parameter ε will become clear in Section 2.6. Additional variations on the forcing include the anisotropic perturbations considered by Rubinstein & Barton (1987) and the colored noise treated by Carati (1990).

Langevin models have often played a useful role in the development of turbulence theories, because they entail replacing the nonlinear interactions with a balance between an effective force $\hat{\mathbf{f}}_e(\hat{\mathbf{k}})$ and an effective viscosity $\nu_e(\hat{\mathbf{k}})$ (e.g. Kraichnan 1971). Thus, one seeks to replace the nonlinear Navier-Stokes equations with the linear equation $-i\omega\hat{\mathbf{u}}(\hat{\mathbf{k}}) = \nu_e(\hat{\mathbf{k}})k^2\hat{\mathbf{u}}(\hat{\mathbf{k}}) + \hat{\mathbf{f}}_e(\hat{\mathbf{k}})$. The force with power-law spectrum $D(k) = D_o k^{4-d-\varepsilon}$ acting at all inertial-range scales $0 < k < \Lambda_o$, where Λ_o is a viscous cutoff, should be interpreted as an effective, or renormalized, force. The renormalized force represents a particular class of nonlinear interactions responsible for the inertial-range cascade. The *Correspondence Principle* of Yakhot & Orszag (1986), largely misunderstood in subsequent literature, models the renormalized force given by Equation 6 with $D(k) = D_o k^{4-d-\varepsilon}$ and $\varepsilon = 4$ in three dimensions. Assuming the existence of a renormalized force from the outset, the purpose of the RG method is to calculate the renormalized viscosity, or more generally the renormalized propagator, as in the analyses of Fournier & Frisch (1983) and Yakhot & Orszag (1986). Recently, Canuto & Dubovikov (1996) also emphasized that $D(k) = D_o k^{-3}$ acting in three dimensions at all inertial-range scales should be interpreted as the renormalized forcing spectrum. Based on Wyld's (1961) RPT diagrams and locality assumptions, they developed turbulence models involving an effective

force with $D(k) \propto k^{-3}$ and an effective viscosity. Lam (1992) presented a different view of the force, one that is problematic because the self-similarity of the inertial range is sacrificed.

2.2 *The Renormalization-Group Scale-Removal Procedure*

2.2.1 SEPARATION OF SCALES The goal of the RG procedure is to calculate the effect of high-wavenumber, high-frequency, inertial-range interactions on the low-wavenumber, low-frequency velocity field, and so to derive an effective, closed equation for the velocity $\hat{\mathbf{u}}(\hat{\mathbf{k}})$ in the limit $\hat{\mathbf{k}} \to 0$. This is done by partitioning wavenumber space into a series of nested shells and iteratively evaluating the effect of the outer-most shell on the remainder. Before removal of the first shell, the highest wavenumber is the viscous cutoff $\Lambda = \Lambda_o$, the viscosity is $\nu(\Lambda_o) = \nu_o$, and the propagator is the bare propagator $G(\hat{\mathbf{k}}) = G^{(0)}(\hat{\mathbf{k}})$.

At an arbitrary point in the scale-removal procedure, the upper cutoff wavenumber is simply denoted Λ, the effective viscosity is $\nu(\Lambda)$, and the propagator is $G(\hat{\mathbf{k}}) = (-i\omega + \nu(\Lambda)k^2)^{-1}$. There is no vertex renormalization in the lowest-order RG analysis, and therefore $\Gamma_{imn}(\hat{\mathbf{k}}, \hat{\mathbf{p}}, \hat{\mathbf{q}}) = \Gamma^{(0)}_{imn}(\hat{\mathbf{k}}, \hat{\mathbf{p}}, \hat{\mathbf{q}})$. For removal of the next shell $[\Lambda - \Delta\Lambda, \Lambda]$, the velocity is decomposed into

$$\hat{u}_i^>(\hat{\mathbf{k}}) \equiv \hat{u}_i(\hat{\mathbf{k}}), \quad \Lambda - \Delta\Lambda < k < \Lambda; \quad \hat{u}_i^>(\hat{\mathbf{k}}) \equiv 0, \quad k < \Lambda - \Delta\Lambda \tag{7}$$

$$\hat{u}_i^<(\hat{\mathbf{k}}) \equiv 0, \quad \Lambda - \Delta\Lambda < k < \Lambda; \quad \hat{u}_i^<(\hat{\mathbf{k}}) \equiv \hat{u}_i(\hat{\mathbf{k}}), \quad k < \Lambda - \Delta\Lambda. \tag{8}$$

The RG procedure involves the elimination of $\hat{u}_i^>$ from the equation for $\hat{u}_i^<$. Therefore the equations of motion (Equation 3) are separated according to the definitions in Equation 7 and Equation 8 into an equation for modes outside the shell,

$$\begin{aligned}\hat{u}_i^<(\hat{\mathbf{k}}) = {} & G(\hat{\mathbf{k}})\,\hat{f}_i^<(\hat{\mathbf{k}}) - \frac{i\lambda_o}{2} G(\hat{\mathbf{k}})\; P_{imn}(\mathbf{k}) \\ & \times \int \frac{d\hat{\mathbf{p}}\,d\hat{\mathbf{q}}}{(2\pi)^{d+1}} \big(\hat{u}_m^<(\hat{\mathbf{q}})\hat{u}_n^<(\hat{\mathbf{p}}) + 2\hat{u}_m^<(\hat{\mathbf{q}})\hat{u}_n^>(\hat{\mathbf{p}}) \\ & + \hat{u}_m^>(\hat{\mathbf{q}})\hat{u}_n^>(\hat{\mathbf{p}})\big)\delta(\hat{\mathbf{k}} - \hat{\mathbf{p}} - \hat{\mathbf{q}}),\end{aligned} \tag{9}$$

and an equation for modes inside the shell,

$$\begin{aligned}\hat{u}_i^>(\hat{\mathbf{k}}) = {} & G(\hat{\mathbf{k}})\,\hat{f}_i^>(\hat{\mathbf{k}}) - \frac{i\lambda_o}{2} G(\hat{\mathbf{k}})\; P_{imn}(\mathbf{k}) \\ & \times \int \frac{d\hat{\mathbf{p}}\,d\hat{\mathbf{q}}}{(2\pi)^{d+1}} \big(\hat{u}_m^<(\hat{\mathbf{q}})\hat{u}_n^<(\hat{\mathbf{p}}) + 2\hat{u}_m^<(\hat{\mathbf{q}})\hat{u}_n^>(\hat{\mathbf{p}}) \\ & + \hat{u}_m^>(\hat{\mathbf{q}})\hat{u}_n^>(\hat{\mathbf{p}})\big)\delta(\hat{\mathbf{k}} - \hat{\mathbf{p}} - \hat{\mathbf{q}}).\end{aligned} \tag{10}$$

The parameter $\lambda_o = 1$ is introduced to facilitate the construction of a perturbation expansion, and the forces $\hat{f}_i^>(\hat{\mathbf{k}})$ and $\hat{f}_i^<(\hat{\mathbf{k}})$ are defined analogously to Equation 7 and Equation 8. The two-point correlation function of the force is Equation 6, with $D(k) = D_o k^{4-d-\varepsilon}$ for $0 < k < \Lambda$.

2.2.2 THE FORCE AND THE DISTANT-INTERACTION APPROXIMATION The force, as a renormalized quantity, already accounts for some of the nonlinear interactions in the convolution integrals of Equation 9 and Equation 10. For example, some pairs of modes with wavevectors $\mathbf{p}$ and $\mathbf{q}$ of comparable length $p \approx q < \Lambda - \Delta\Lambda$ will force a mode with wavevector $\mathbf{k}$ and $\Lambda - \Delta\Lambda < k < \Lambda$. This suggests that local interactions involving $\hat{u}_m^<(\hat{\mathbf{q}})\hat{u}_n^<(\hat{\mathbf{p}})$ contribute to $\hat{f}_i^>(\hat{\mathbf{k}})$ in Equation 10. These interactions are local in the sense that the constraint $\delta(\hat{\mathbf{k}} - \hat{\mathbf{p}} - \hat{\mathbf{q}})$ implies that the wavenumber k is at most twice the wavenumbers $p \approx q$, i.e. $p \approx q < k < 2p \approx 2q$. Those interactions involving $\hat{u}_m^<(\hat{\mathbf{q}})\hat{u}_n^<(\hat{\mathbf{p}})$ that are nonlocal are not well-represented by Langevin-style forcing of the mode $\hat{u}_i^>(\hat{\mathbf{k}})$: for example, the interaction involving $q \ll \Lambda - \Delta\Lambda$, $p < \Lambda - \Delta\Lambda$, and $\Lambda - \Delta\Lambda < k < \Lambda$ with $p \approx k$. This interaction would instead contribute to a cusp in the effective viscosity acting on $\hat{u}_n^<(\hat{\mathbf{p}})$ for $p \to \Lambda - \Delta\Lambda$ from below (Kraichnan 1976). The interactions leading to a constant contribution to the effective viscosity acting on the large-scale mode $\hat{u}_i^<(\hat{\mathbf{k}})$ are also nonlocal, with $k \ll \Lambda - \Delta\Lambda$ and $\Lambda - \Delta\Lambda < p, q < \Lambda$.

This concept of local and nonlocal interactions is at present qualitative but is central to understanding the RG procedure. To avoid double-counting the local interactions that are attributed to the renormalized force, the convolution integrals should include only the remaining interactions. This is achieved in the RG method by evaluating all integrals in the distant-interaction limit $\hat{\mathbf{k}}/\hat{\mathbf{q}} \to 0$, where $k \ll \Lambda - \Delta\Lambda$ and $\Lambda - \Delta\Lambda < q < \Lambda$ (Kraichnan 1987). Note that the ratio of frequencies approaches zero, in addition to the ratio of wavenumbers. The accuracy with which the force (Equation 6) with $D(k) = D_o k^{4-d-\varepsilon}$ models the physics of the local interactions has not been quantified. Likewise, there is no systematic measure of the accuracy of the distant-interaction approximation for capturing the physics of the remaining interactions.

Even in the absence of the force (for example, in the passive scalar problem treated in Section 3), one might argue that the distant-interaction approximation is a reasonable approximation to deduce the effect of small-scale, high-frequency motions on the large-scale, low-frequency motions. In fact, the distant-interaction approximation would be justified if a spectral gap existed between small and large wavenumbers, but of course this is not the case in the strongly nonlinear turbulence problem. For a continuous spectrum, the errors introduced by the distant-interaction approximation may accumulate with iteration as follows: At each stage of the scale-removal procedure, the distant-interaction approximation excludes the effects of modes with wavenumber

above the cutoff on modes with wavenumbers just below the cutoff. However, at the next iteration, the effective equation for large scales deduced in the distant-interaction limit will be used in the entire wavenumber space below the cutoff, including just below the cutoff. Nevertheless, the simplifications provided by the distant-interaction limit make it attractive as an (uncontrolled) approximation scheme. As shall be seen in Section 3, the distant-interaction approximation yields accurate results for a model passive scalar problem.

2.2.3 ELIMINATION OF THE HIGH-WAVENUMBER MODES Once Equation 9 and Equation 10 are defined, $\hat{u}_i^>$ is expanded in the perturbation series $\hat{u}_i^> = \hat{u}_i^{(0)} + \lambda_o \hat{u}_i^{(1)} + \lambda_o^2 \hat{u}_i^{(2)} + O(\lambda_o^3)$. Note that the renormalized force appears at lowest order and so the local interactions represented by the force dominate the nonlocal interactions, in keeping with traditional turbulence folklore. The expansion for $\hat{u}_i^>$ allows one to eliminate, to any finite order in λ_o, $\hat{u}_i^>$ from Equation 9 in favor of $\hat{u}_i^{(0)} = G\hat{f}_i^>$ and $\hat{u}_i^<$ (Forster et al 1977, Fournier & Frisch 1983, Yakhot & Orszag 1986). Ensemble averaging of the small-scale force $\hat{f}_i^>$ then leads to an equation containing $\hat{u}_i^<$ only (now also a series in powers of λ_o). In this process, large-scale quantities are assumed to be the same for all realizations of the fine-scale ensemble (Smith & Reynolds 1992). Later papers have proposed a scale-elimination procedure based on conditional averaging in an attempt to remove this restriction (McComb & Watt 1990, 1992). Rose (1977) gave another perspective on this issue.

If the Navier-Stokes equations (Equation 3) had been nondimensionalized, the artificial expansion parameter λ_o would be replaced by a local Reynolds number of the modes in the high-wavenumber shell (Yakhot & Orszag 1986, Smith & Reynolds 1992, Yakhot & Smith 1992). Based on the length scale $l = 1/\Lambda$, time scale $l^2/\nu(\Lambda)$, and velocity scale $D_o^{1/2}(l^{\varepsilon-2}/\nu(\Lambda))^{1/2}$, the local Reynolds number $\bar{\lambda}$ may be defined by

$$\bar{\lambda}^2 = \frac{\lambda_o^2 D_o}{\nu^3(\Lambda)\Lambda^\varepsilon}, \tag{11}$$

where the dimensions of D_o are $[D_o] = L^{6-\varepsilon}T^{-3}$, following from Equation 6 with $D(k) = D_o k^{4-d-\varepsilon}$. It will be seen that the RG method relies on a low-order truncation of the perturbation expansion for $\hat{u}_i^>$ in powers of $\bar{\lambda}$. For $\varepsilon > 0$, Λ large and $\nu(\Lambda)$ small, $\bar{\lambda} \propto \Lambda^{-\varepsilon}$ is small compared with the global Reynolds number $Re \equiv U_o L_o/\nu_o$, where U_o and L_o are, respectively, velocity and length scales characteristic of the large-scale flow. Because it will be found that $\nu(\Lambda)$ increases as Λ decreases, the value of $\bar{\lambda}$ remains small compared to Re (although may not be less than unity) as wavenumber shells are eliminated. Thus the expansion in powers of $\bar{\lambda}$ is at least more plausible than an expansion in powers of Re.

One iteration of the RG scale-elimination procedure will now be illustrated using diagrams.

2.3 *Diagrams for One Iteration of the Renormalization-Group Procedure*

In this section, we develop a diagrammatic expansion for the velocity $\hat{u}_i^<(\hat{\mathbf{k}})$ of wavenumbers below the cutoff Λ, paralleling the RG procedure developed by Forster et al (1977), Fournier & Frisch (1983), and Yakhot & Orszag (1986). The diagrams are simply a compact notation for representing the terms in the series for $\hat{u}_i^<(\hat{\mathbf{k}})$. We hope that the diagrammatic presentation of the RG scale-elimination procedure will help clarify some features of the method, and also help relate the RG method to RPT theories in general and to Kraichnan's (1959) DIA in particular (Section 2.8). Wyld (1961) used diagrams to facilitate the construction of three coupled integral equations for the Fourier transform $\Phi_{ij}(\hat{\mathbf{k}})$ of the two-point correlation function $R_{ij}(\mathbf{x}-\mathbf{x}', |t-t'|)$, the renormalized propagator $G(\hat{\mathbf{k}})$, and the renormalized vertex operator $\Gamma_{imn}(\hat{\mathbf{k}}, \hat{\mathbf{p}}, \hat{\mathbf{q}})$. A low-order truncation of these coupled integral equations is equivalent to Kraichnan's DIA. Wyld's rules for constructing diagrams are known to be flawed at orders higher than DIA (Lee 1965, Martin et al 1973). Vertex corrections are rarely retained in turbulence closures because of their complexity and because they vanish in the limit $k \to 0$ as a result of Galilean invariance (Forster et al 1977). There is no vertex renormalization in DIA, and as is seen below, there is no vertex renormalization in the lowest-order RG analysis.

Dropping all subscripts and arguments, let us symbolically represent $\hat{f}^<$ by a cross, $\hat{u}^<$ by a solid line, and the lowest order contribution to $\hat{u}^>$, namely $\hat{u}^{(0)} = G\hat{f}^>$, by a crossed solid line. We also denote the propagator G by a dashed line and the vertex Γ by a circle (Figure 1). Figure 2 shows the unaveraged equation for $\hat{u}^<$ found by keeping terms to $O(\lambda_o)$ in the series expansion for $\hat{u}^>$. In each diagram, the number of circles is also the order of λ_o and can be used to keep track of the order of the perturbation expansion for $\hat{u}^>$:

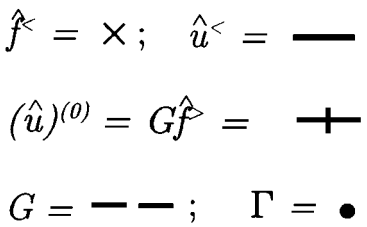

Figure 1 Symbols for the diagrammatic representation of $\hat{u}^<$.

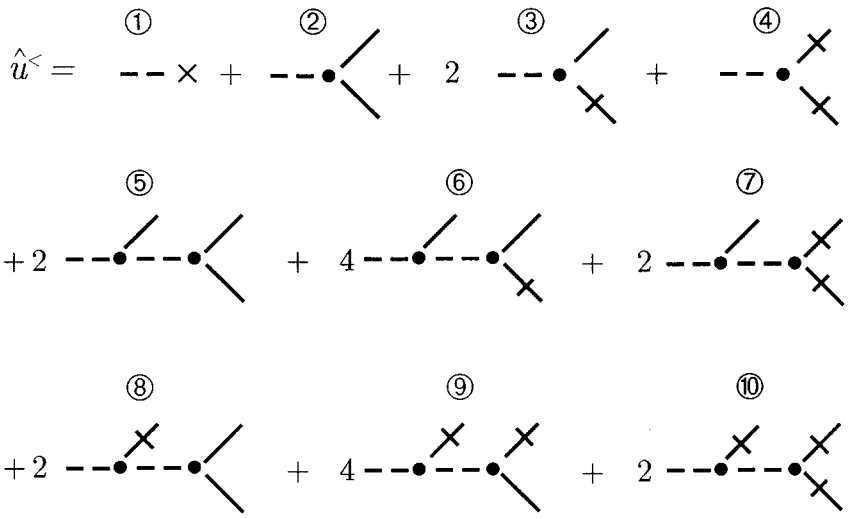

Figure 2 The unaveraged series for $\hat{u}^<$ to $O(\lambda_o^2)$.

One circle indicates a term $O(\lambda_o^0)$ in the expansion for $\hat{u}^>$; two circles indicate a term $O(\lambda_o)$ in the expansion for $\hat{u}^>$, and so on.

Rules for constructing the diagrams are as follows (see Figure 2): (*a*) Three lines meet at each circle, one incoming line and two outgoing lines, representing G, $\hat{u}^<$, and $\hat{u}^{(0)} = G\hat{f}^>$; (*b*) the incoming line must be dashed (the G in front of the integral in Equation 10); and (*c*) the three vector arguments of the lines meeting at each circle (the arguments of G, $\hat{u}^<$, or $\hat{u}^{(0)}$) must add up to zero—a result of the delta function in the vertex (Equation 5). The factors of two arise because symmetrical diagrams contribute identically to the equation for $\hat{u}^<$. The diagrammatic representation of $\hat{u}^<$ to any order m in λ_o consists of the diagram representing $G\,\hat{f}^<$, together with all combinations of diagrams with the number of circles less than or equal to m, constructed according to the above rules.

Diagram *1* in Figure 2 represents $G\,\hat{f}^<$, modeling local interactions. All of the remaining diagrams are restricted to include only nonlocal interactions and are evaluated in the distant-interaction limit. Diagram *2* in Figure 2 is $G\Gamma\hat{u}^<\hat{u}^<$. Diagram *4* in Figure 2 represents a zero-average correction to the force with spectrum $D_b(k) \propto k^2$ (Yakhot & Orszag 1986). The subscript b is used to label the correction to the force because it represents the backscatter of energy from small to large scales (Kraichnan 1976, Piomelli et al 1991). At any stage of the scale-elimination procedure, the backscatter force with $D_b(k) \propto k^2$ is small compared with the effective force with $D(k) \propto k^{-3}$ ($d = 3$, $\varepsilon = 4$) in

the distant-interaction limit ($k/\Lambda \to 0$) but is not negligible for $k \to \Lambda - \Delta\Lambda$ (see Section 4).

Diagram *5* in Figure 2 is a cubic nonlinearity, generated by substituting the $\hat{u}^< \hat{u}^<$ term in Equation 10 into the $\hat{u}^< \hat{u}^>$ term in Equation 9. Recall that the cusp behavior of the eddy viscosity arises from the $\hat{u}^< \hat{u}^>$ term in Equation 9 for $k \to \Lambda - \Delta\Lambda$ from below. The cubic nonlinearity was dropped from the effective equation for $\hat{u}^<$ by Yakhot & Orszag (1986) without proper justification (Eyink 1994, McComb 1990). However, subsequent work indicates that retaining the cubic term leads to unphysical results for the eddy viscosity (further discussion of these results is postponed until Section 2.7). Therefore, we shall also drop the cubic term from the remaining analysis.

Diagram *9* of Figure 2, linear in $\hat{u}^<$, represents a correction to the propagator G, and all other diagrams are zero after averaging over the small-scale force. It is easy to reconstruct the actual integral contribution to the equation for $\hat{u}^<$ from its diagram. For example, the rules tell us that Diagram *9* corresponds to

$$4\left(\frac{-i\lambda_o}{2}\right)^2 G(\hat{\mathbf{k}}) P_{imn}(\mathbf{k}) \int \frac{d\hat{\mathbf{p}} d\hat{\mathbf{q}}}{(2\pi)^{d+1}} \delta(\hat{\mathbf{k}} - \hat{\mathbf{p}} - \hat{\mathbf{q}}) G(\hat{\mathbf{q}})\, \hat{f}^>_m(\hat{\mathbf{q}}) G(\hat{\mathbf{p}}) P_{n\alpha\beta}(\mathbf{p}) \times \int \frac{d\hat{\mathbf{r}} d\hat{\mathbf{s}}}{(2\pi)^{d+1}} \delta(\hat{\mathbf{p}} - \hat{\mathbf{r}} - \hat{\mathbf{s}}) \hat{u}^<_\alpha(\hat{\mathbf{r}}) G(\hat{\mathbf{s}})\, \hat{f}^>_\beta(\hat{\mathbf{s}}). \tag{12}$$

The elimination of the shell is completed by averaging over different realizations of the small-scale force $\hat{f}^>$, remembering that $\hat{u}^<$ is assumed to be independent of the fine-scale ensemble. In terms of the diagrams, this averaging is achieved by the following rules: (*a*) All diagrams consisting only of dashed lines, solid lines and circles are discarded—this rule eliminates the cubic and higher-order nonlinearities; (*b*) all diagrams with an odd number of crossed solid lines are also discarded, because they average to zero by the Gaussian nature of the force; (*c*) from diagrams with even numbers of crossed solid lines, new diagrams are generated by connecting all possible pairs of crossed solid lines with a wavy line—this is the operation of averaging, and the wavy lines represent $GG\,\langle \hat{f}^> \hat{f}^> \rangle$; and (*d*) after connecting crossed solid lines, all diagrams are discarded involving a wavy line that can be split into two pieces by severing any single line—this eliminates the remaining zero-average terms.

Figure 3 shows the averaged equation for $\hat{u}^<$ to $O(\lambda_o^2)$, the lowest order involving a nontrivial correction arising from the scale removal. Diagrams *1* and *2* in Figure 3 are, respectively, $G\,\hat{f}^<$ and $G\Gamma\hat{u}^<\hat{u}^<$ and are unaffected by the small-scale ensemble averaging. Diagram *3* corresponds to the average of Equation 12 above, representing a correction to the propagator. Using the definition of the force correlation (Equation 6) and the spectrum $D(k) = D_o k^{4-d-\varepsilon}$,

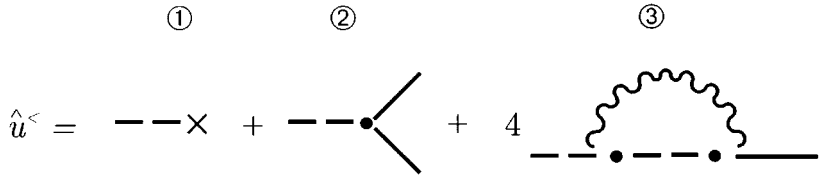

Figure 3 The averaged series for $\hat{u}^<$ to $O(\lambda_o^2)$.

this term is rewritten as

$$-G(\hat{\mathbf{k}})\hat{u}_i^<(\hat{\mathbf{k}})k^2 I_\nu(\hat{\mathbf{k}}) \equiv -G(\hat{\mathbf{k}})\hat{u}_\alpha^<(\hat{\mathbf{k}}) \cdot 2\lambda_o^2 D_o P_{imn}(\mathbf{k}) \int \frac{d\hat{\mathbf{q}}}{(2\pi)^{d+1}}$$
$$\times G(\hat{\mathbf{q}})G(-\hat{\mathbf{q}})G(\hat{\mathbf{k}}-\hat{\mathbf{q}})P_{n\alpha\beta}(\mathbf{k}-\mathbf{q})P_{m\beta}(\mathbf{q})q^{4-d-\varepsilon}, \qquad (13)$$

where the eddy damping factor $k^2 I_\nu(\hat{\mathbf{k}})$ is a correction to $G^{-1}(\hat{\mathbf{k}})$, with k^2 factored out for convenience; it will be found that $I_\nu(\hat{\mathbf{k}})$ is a constant contribution to the eddy viscosity in the distant-interaction limit $\hat{\mathbf{k}}/\hat{\mathbf{q}} \to 0$ (Section 2.4). In Equation 13, the frequency integration domain is infinite, but the spatial integration domain is restricted to the intersection of the high-wavenumber shells $\Lambda - \Delta\Lambda < q$, $|\mathbf{k}-\mathbf{q}| < \Lambda$.

Even though the series for $\hat{u}_i^<(\hat{\mathbf{k}})$ is truncated at $O(\lambda_o^2)$ in the lowest-order RG analysis, for comparison with RPT it is instructive to consider the higher-order contributions generated by the diagrammatic method described above; the scale-elimination procedure symbolically produces the same corrections to the vertex at $O(\lambda_o^3)$ and to the propagator at $O(\lambda_o^4)$ as Wyld's procedure (Figures 3 and 4 in Wyld 1961). One difference between the RG and Wyld diagrams is the implied domains of integration for the convolution integrals involving correlation functions. In the Wyld expansion, the velocity correlations are integrated over the entire domain with dummy wavenumbers of integration in the range $[0, \infty]$. In the RG diagrams, the force correlations are integrated over a region for which dummy wavenumbers are restricted to high-wavenumber shells.

2.4 *Evaluation of the Renormalized Propagator*

At an arbitrary point in the scale-removal procedure, where the cutoff is Λ and the viscosity is $\nu(\Lambda)$, the correction to the propagator from the eliminated shell is found from Equation 13 above. Yakhot & Orszag (1986) calculated $I_\nu(\hat{\mathbf{k}})$ to lowest nontrivial order in the distant-interaction limit $\hat{\mathbf{k}}/\hat{\mathbf{q}} \to 0$. The spatial integration domain in Equation 13 is the intersection of the domains $\Lambda - \Delta\Lambda < q < \Lambda$ and $\Lambda - \Delta\Lambda < |\mathbf{k}-\mathbf{q}| < \Lambda$. Because this intersection reduces to a spherical shell at lowest order in the distant-interaction limit (because $k \ll q$),

the calculation is simplified if the integrand is symmetrized about the average radius of the shell $q' = q - k/2$. The lowest-order contribution to the integral is then calculated by expanding the integrand in powers of $\hat{\mathbf{k}}/\hat{\mathbf{q}}'$, keeping terms to lowest nontrivial order, and integrating over the spherical shell $\Lambda - \Delta\Lambda < q' < \Lambda$. In this case, the $O((k/q')^0)$ contribution integrates to zero because of symmetries, and the lowest nontrivial order in $\hat{\mathbf{k}}/\hat{\mathbf{q}}'$ corresponds to $O((\omega/\Omega')^0)$ and $O(k/q')$, where $\hat{\mathbf{k}} = (\mathbf{k}, \omega)$ and $\hat{\mathbf{q}}' = (\mathbf{q}', \Omega')$. The frequency integration is performed using contour-integral methods, and after some algebra, one finds

$$I_\nu(\hat{\mathbf{k}}) = \frac{\lambda_o^2}{\nu^2(\Lambda)} \frac{D_o S_d}{(2\pi)^d} \frac{(d^2 - d - \varepsilon)}{2d(d+2)} \frac{(\Lambda - \Delta\Lambda)^{-\varepsilon} - \Lambda^{-\varepsilon}}{\varepsilon} \equiv \Delta\nu, \tag{14}$$

where S_d is the area (nondimensionalized) of a unit sphere in d-dimensions. The correction to the inverse propagator $G^{-1}(\hat{\mathbf{k}})$ is thus simply $k^2\Delta\nu$, with $\Delta\nu$ a constant correction to the viscosity (independent of $\hat{\mathbf{k}}$) in the distant-interaction limit. The renormalized viscosity $\nu(\Lambda - \Delta\Lambda)$ after removal of the shell $[\Lambda - \Delta\Lambda, \Lambda]$ is given by $\nu(\Lambda - \Delta\Lambda) = \nu(\Lambda) + \Delta\nu$.

2.5 *Iteration of the Renormalization-Group Procedure Using Differential Relations*

Sections 2.3 and 2.4 show that, after removal of the smallest length and time scales, the only correction to Equation 9 is a correction to the effective viscosity. Thus the effective equation for motion at larger scales and longer times has the same form as the original equation, with a renormalized coefficient, suggesting that the scale-removal procedure may be iterated to yield effective equations for larger scales and longer times. After replacing $\hat{u}_i^<(\hat{\mathbf{k}})$ by $\hat{u}_i(\hat{\mathbf{k}})$, the shell-elimination procedure may be repeated, starting from Equation 7 and Equation 8. In practice, differential relations may be derived for the calculation of the renormalized viscosity after many iterations.

From Equation 14, in the limit $\Delta\Lambda \to 0$, the differential equation for $\nu(\Lambda)$ is

$$\frac{d\nu(\Lambda)}{d\Lambda} \equiv \lim_{\Delta\Lambda\to 0} \frac{\Delta\nu}{\Delta\Lambda} = -\frac{\tilde{A}_d S_d}{(2\pi)^d} \frac{\nu(\Lambda)\bar{\lambda}^2(\Lambda)}{\Lambda}, \tag{15}$$

where $\tilde{A}_d = (d^2 - d - \varepsilon)/(2d(d+2))$ following the notation of Yakhot & Orszag (1986), and the nondimensional expansion parameter $\bar{\lambda}(\Lambda)$ (the effective Reynolds number) is given by Equation 11.

Iteration of the scale-removal process is carried out by integration of Equation 15, subject to the initial condition $\nu(\Lambda_o) = \nu_o$, leading to

$$\nu(\Lambda) = \nu_o \left(1 + \frac{3\tilde{A}_d}{\nu_o^3} \frac{D_o S_d}{(2\pi)^d} \frac{\Lambda^{-\varepsilon} - \Lambda_o^{-\varepsilon}}{\varepsilon}\right)^{1/3}, \tag{16}$$

where $\lambda_o = 1$ has been used. A similar expression for $\bar{\lambda}(\Lambda)$ follows by substituting Equation 16 into Equation 11. The solution for large scales is found for $\Lambda \ll \Lambda_o$, where (Yakhot & Orszag 1986)

$$\nu(\Lambda) \sim \left(\frac{3\tilde{A}_d}{\varepsilon}\right)^{1/3} \left(\frac{S_d D_o}{(2\pi)^d}\right)^{1/3} \Lambda^{-\varepsilon/3}; \quad \bar{\lambda} \sim \varepsilon^{1/2} \left(\frac{3\tilde{A}_d S_d}{(2\pi)^d}\right)^{-1/2}. \tag{17}$$

The relations of Equation 17 show that, for $\varepsilon > 0$, the renormalized Reynolds number approaches a constant and the renormalized viscosity increases and dominates the molecular viscosity as the cutoff Λ decreases. Both $\nu(\Lambda)$ and $\bar{\lambda}$ are independent of the dissipation-range quantities Λ_o and ν_o, as expected for inertial-range coefficients.

2.6 *The ε Expansion*

From Equation 17, the nondimensional coupling constant $\bar{\lambda} \sim \varepsilon^{1/2}$ for $\Lambda \ll \Lambda_o$. This means that in the limit $\varepsilon \to 0$, (*a*) the $\bar{\lambda}$ expansion can be rewritten as an ε expansion, and (*b*) the $O(\lambda_o^3)$ terms not included in Figure 3 are negligible. Because corrections to the propagator arise at even orders in $\bar{\lambda}$, and are simply corrections to the viscosity in the distant-interaction limit, the differential equation for $\nu(\Lambda)$ (Equation 15) may be written as

$$\begin{aligned} \frac{d\nu(\Lambda)}{d\Lambda} &= -\left(A_d^0 + \varepsilon A_d^1\right) \frac{S_d}{(2\pi)^d} \frac{\nu(\Lambda)\bar{\lambda}^2(\Lambda)}{\Lambda} + O(\bar{\lambda}^4) \\ &= -\left(A_d^0 + \varepsilon A_d^1 + O(\varepsilon)\right) \frac{S_d}{(2\pi)^d} \frac{\nu(\Lambda)\bar{\lambda}^2(\Lambda)}{\Lambda}, \end{aligned} \tag{18}$$

where the last equality follows from dimensional analysis. The definitions $A_d^0 = (d^2 - d)/(2d(d+2))$ and $A_d^1 = -1/(2d(d+2))$ follow from the definition of $\tilde{A}_d$.

From Equation 18, at each iteration of the scale removal, the nondimensional coefficient is known only to $O(1)$ unless the $O(\bar{\lambda}^4)$ terms are retained. In other words, the term proportional to εA_d^1 must be dropped from Equation 18 for a consistent asymptotic expansion at each iteration step. The asymptotically consistent solutions for $\nu(\Lambda)$ and $\bar{\lambda}$ are then given by Equation 17, with $\tilde{A}_d$ replaced by A_d^0. Because the factor $\Lambda^{-\varepsilon/3}$ is (*a*) dimensional and (*b*) common to all orders in the $\bar{\lambda}$ expansion, it should not be expanded in powers of ε. Even had the problem been nondimensionalized from the start, a Taylor-series expansion of the nondimensional cutoff raised to the power ε would be invalid owing to the appearance of divergent logarithms as $\Lambda \to 0$ (Woodruff 1994).

Yakhot & Orszag (1986) extended the asymptotic expansion for $\varepsilon \to 0$ to the case $\varepsilon = 4$, corresponding to the Kolmogorov energy spectrum in $d = 3$ dimensions. Although the results are extrapolated from a valid asymptotic

expansion, the only justification for the ε expansion at $\varepsilon = 4$ is the fact that it leads to good agreement for turbulence coefficients such as the Kolmogorov constant, and good predictions for turbulence models such as the $\mathcal{K} - \mathcal{E}$ model (Section 4). This agreement is intriguing but difficult to interpret given the assumptions and uncontrolled approximations of the method (Kraichnan 1987, Woodruff 1994).

Even in the case $\varepsilon \to 0$, when the truncation error at each iteration stage is small, the error accumulated with iteration is not known to remain small. In some problems, one can show that accumulated error remains small, and in these problems the neglected higher-order terms are called *irrelevant variables* (Wilson 1975, Eyink 1994). However, as discussed by Eyink (1994), the neglected cubic and higher-order nonlinearities appearing at higher order in $\bar{\lambda}$ are not irrelevant in the sense of Wilson (1975), and at present, there is no estimate for the build-up of error for either $\varepsilon \to 0$ or $\varepsilon = 4$.

To eliminate all wavenumbers up to $k \ll \Lambda_o$ in the inertial range, the cutoff Λ is taken at $\Lambda = k$. Notice that this again violates the distant-interaction approximation with $k/\Lambda \to 0$ (see Section 2.2.2). The velocity $\hat{u}_i^<(\hat{\mathbf{k}})$ is redefined as $\hat{u}_i(\hat{\mathbf{k}})$, and the effective equation for large inertial-range lengths and times becomes

$$\hat{u}_i(\hat{\mathbf{k}}) = G(\hat{\mathbf{k}})\,\hat{f}_i(\hat{\mathbf{k}}) - \frac{i}{2}G(\hat{\mathbf{k}})P_{imn}(\mathbf{k})\int \frac{d\hat{\mathbf{q}}d\hat{\mathbf{p}}}{(2\pi)^{d+1}}\,\hat{u}_m(\hat{\mathbf{q}})\hat{u}_n(\hat{\mathbf{p}})\delta(\hat{\mathbf{k}} - \hat{\mathbf{p}} - \hat{\mathbf{q}}), \tag{19}$$

where $G(\hat{\mathbf{k}}) = (-i\omega + \nu(k)k^2)^{-1}$, and

$$\nu(k) = \left(\frac{3A_d^0}{\varepsilon}\right)^{1/3}\left(\frac{S_d D_o}{(2\pi)^d}\right)^{1/3} k^{-\varepsilon/3}. \tag{20}$$

In Equation 19, the force may be discarded and local interactions may be restored in the convolution integral. The parameter D_o proportional to the energy input rate is as yet unknown. Because the energy input rate must be equal to the energy dissipation rate ε in a statistically steady state, the equation for the energy following from Equation 19 may be used to find D_o in terms of ε (Section 4).

2.7 *The Cubic Nonlinearity*

Upon iteration of the scale-removal procedure, the cubic term generates additional contributions to the eddy viscosity, different from the contribution (Equation 14) generated by the quadratic nonlinearity. Carati (1991) stated that the cubic term's contribution is negligible in the distant-interaction limit $k/\Lambda \to 0$ when $k/\Delta\Lambda \to 0$. However, a reexamination by Smith et al (1991) showed that the cubic term's contribution is independent of the shell width $\Delta\Lambda$ for $k/\Lambda \to 0$ when $k/\Delta\Lambda \to 0$ followed by $\Delta\Lambda/\Lambda \to 0$. This means that

the cumulative contribution to the eddy viscosity generated by the cubic term would diverge after the removal of many infinitesimal shells. By contrast, the change in the eddy viscosity arising from the quadratic nonlinearity is proportional to $\Delta\Lambda$, such that a differential equation for $dv/d\Lambda$ may be defined in the limit $\Delta\Lambda/\Lambda \to 0$ (Equation 14 and Equation 15).

Zhou et al (1988, 1989) pursued a numerical version of the RG scale-elimination procedure using finite shells. They abandoned the distant-interaction approximation and argued that the cubic nonlinearity should be retained. Although their numerical version of the RG method recovers the cusp behavior of the eddy viscosity for $k \to \Lambda - \Delta\Lambda$, the eddy viscosity depends strongly on the width $\Delta\Lambda$ of the eliminated wavenumber shells and is negative for small values of $\Delta\Lambda$ (Carati 1991). To address these problems, Zhou et al (1993) suggested a partial averaging method, while Carati (1997) proposed an iterative-filtering technique.

Thus we conclude that (*a*) the distant-interaction approximation is a crucial part of the present formulation, (*b*) the cubic nonlinearity leads to unphysical results for the eddy viscosity, and (*c*) the cusp in the eddy viscosity is not currently accessible by this method.

2.8 *The Yakhot-Orszag Method, Kraichnan's Direct Interaction Approximation, and Related Renormalized Perturbation Theories*

One would like to isolate and understand each of the approximations used in the Yakhot-Orszag analysis, including (*a*) the renormalized force to model local interactions, (*b*) the distant-interaction approximation, (*c*) the extrapolation of the ε expansion to $\varepsilon = 4$, and (*d*) the error accumulated with iteration (assumed to be negligible). In this section, we summarize a few efforts in this direction, mainly using Kraichnan's (1959) direct interaction approximation (DIA). In addition, Section 3 discusses in detail the work of Avellaneda & Majda (1990, 1992a,b, 1994) dealing with a model problem of passive scalar convection. Their problem has the advantages that no force and no ε expansion need be invoked, and the results can be compared with exact long-time, large-scale solutions. Therefore, for a problem related to Navier-Stokes turbulence, the accuracy of the iterative scale removal together with the distant-interaction approximation may be assessed.

As mentioned in Section 2.3, the RG diagrams for one iteration of the scale-removal procedure have a one-to-one correspondence with the low-order Wyld diagrams for the bare perturbation series of the propagator and the vertex. Furthermore, iteration of the RG procedure is closely related to resummation of infinite subsets of terms in the bare RPT series. In particular, Kraichnan's DIA (1959) can be formulated as a low-order RPT (e.g. Phythian 1969, Martin et al 1973) and is based on a renormalized propagator given symbolically by the

same "one-loop" diagram (Diagram *3*) appearing in Figure 3 (see also Figure 9 in Wyld 1961). However, one must be careful when making such an analogy between the RG and the DIA closures, because there are important differences between their respective renormalized variables, such as the domains of integration implied by the diagrams and the specific form of the renormalized force correlation implied by the wavy lines. There are, nevertheless, a few remarks that we can make regarding the relationship implied by the one-loop structure of the RG and DIA closures.

It is helpful in understanding that relationship to compare the propagators in the two formulations. The DIA propagator is defined by the equation $G(\hat{\mathbf{k}}) = (-i\omega + \nu_o k^2 + \eta(\hat{\mathbf{k}}))^{-1}$ (Kraichnan 1959), where the eddy damping $\eta(\hat{\mathbf{k}})$ may be written

$$\eta(\hat{\mathbf{k}}) = 2P_{imn}(\mathbf{k}) \int \frac{d\hat{\mathbf{q}}}{(2\pi)^{d+1}} G(\hat{\mathbf{q}}) G(-\hat{\mathbf{q}}) G(\hat{\mathbf{k}} - \hat{\mathbf{q}}) \times P_{ni\alpha}(\mathbf{k} - \mathbf{q}) P_{m\alpha}(\mathbf{q}) \tilde{D}(\hat{\mathbf{q}}). \tag{21}$$

$\tilde{D}(\hat{\mathbf{k}})$ is the correlation of the DIA renormalized force fixed by

$$\tilde{D}(\hat{\mathbf{k}}) \propto D(\hat{\mathbf{k}}) + \int \frac{d\hat{\mathbf{q}}}{(2\pi)^{d+1}} P_{imn}(\mathbf{k}) P_{i\alpha\beta}(\mathbf{k}) P_{m\beta}(\mathbf{q}) P_{n\alpha}(\mathbf{k} - \mathbf{q}) \times G(\hat{\mathbf{q}}) G(-\hat{\mathbf{q}}) \tilde{D}(\hat{\mathbf{q}}) G(\hat{\mathbf{k}} - \hat{\mathbf{q}}) G(-\hat{\mathbf{k}} + \hat{\mathbf{q}}) \tilde{D}(\hat{\mathbf{k}} - \hat{\mathbf{q}}) \tag{22}$$

and $D(\hat{\mathbf{k}})$ is the correlation of an external random force. Note that here the integrals are over all wavenumber space. While the renormalized force correlation $\tilde{D}(\hat{\mathbf{k}})$ does not in general have the RG form, if we nevertheless make the substitution $\tilde{D}(\hat{\mathbf{k}}) = D_o k^{4-d-\varepsilon}$, the expression for the DIA eddy damping (Equation 21) is formally identical to the RG expression for the eddy damping, $k^2 I_\nu(\hat{\mathbf{k}})$ in Equation 13. When suitable approximations are applied to the DIA eddy damping, the DIA and RG propagators become identical.

One such approximation is the DSTA of Kraichnan (1987), a precise formulation of the distant-interaction approximation in terms of an adjustable parameter β describing the nonlocality of triad interactions. Applying this approximation to a class of statistical closures including the DIA, Kraichnan found β-dependent expressions for quantities like the Kolmogorov constant that are quite close to the RG values and that are fairly insensitive to the choice of β. Besides demonstrating the commonality of the RG and DIA statistical closures, this work highlights the central role of the distant-interaction approximation in the RG analysis.

The ε expansion may also be applied to the DIA equations; this is particularly convenient if a similarity solution for the DIA equations is sought (Woodruff 1994). Because the RG model force with an ε-dependent correlation is not required in the DIA formulation, ε appears as a parameter in the similarity

variables that is not fixed by the DIA equations. When expanded in ε, the DIA integral equations become simple algebraic equations whose solution to leading order coincides with the RG solution. In the DIA, the distant-interaction approximation in wavenumber space follows from the ε expansion because the integrals involved are dominated as $\varepsilon \to 0$ by contributions from triads for which $k \ll q$. These ideas were pursued further in Woodruff (1995), where a singular-perturbation analysis of the DIA equations for small ε yields more general, non-self–similar solutions.

In his original work on the DIA, Kraichnan (1959) showed that the DIA predicts a $k^{-3/2}$ inertial-range spectrum instead of the Kolmogorov $k^{-5/3}$ spectrum. He identified the cause of this discrepancy as the inability of DIA to distinguish between the sweeping and straining of small eddies by large eddies. Only straining should affect the small-scale dynamics, but theories based on Eulerian correlations introduce unphysical effects of sweeping by the large eddies (Rose & Sulem 1978). This spurious sweeping, in turn, introduces an additional dimensional parameter that alters the results of the Kolmogorov dimensional analysis. The RG analysis does not include sweeping effects, because apart from the model force, any wavenumber k is influenced only by modes in eliminated shells at higher wavenumbers. Consequently, the RG analysis gives the Kolmogorov spectrum in spite of its relationship with DIA (see also Nelkin & Tabor 1990). Further developments by Kraichnan to eliminate the spurious effect of sweeping in DIA include the explicit removal of the problem-causing interactions (Kraichnan 1964) and various Lagrangian-based RPTs (Kraichnan 1965, 1966). Kraichnan determined an excellent value for Kolmogorov's constant from a Lagrangian RPT, but the theory fails in other respects (Kraichnan 1968). In a related approach to eliminate sweeping from RPTs, L'vov & Procaccia (1995) constructed an RPT for turbulence involving a difference of velocities. In a series of papers, they developed a physical space and time analysis, including nonperturbative effects to account for deviations from Kolmogorov scaling.

3. A MODEL PASSIVE-SCALAR PROBLEM

3.1 *Rigorous Solution in the Long-Time, Large-Distance Limit*

To better understand the mode-elimination RG procedure, we look next at a simpler problem proposed by Avellaneda & Majda (1990). The application of the RG technique to this problem is free of many of the conceptual complications encountered in the previous section, such as the renormalization of the force, the reliance on the ε expansion, and so on. It is thus a purer application of the RG technique and gives a better picture of the fundamentals of the method.

The passive scalar model problem of Avellaneda & Majda (1990) involves a two-dimensional scalar field, $T(t, x, y)$, convected by a parallel flow whose single nonzero velocity component, $v(t, x)$, is random. The scalar field thus satisfies the equation

$$\frac{\partial T}{\partial t} + v(t, x)\frac{\partial T}{\partial y} = \kappa_o \nabla^2 T, \tag{23}$$

where κ_o is the diffusivity of the scalar T. An initial condition $T(t = 0, x, y) = T_o(x, y)$ is specified and the statistics of the random function $v(t, x)$ are taken to be Gaussian with zero mean. The statistics are consequently completely defined by the two-point correlation, $\langle v(t, x)\ v(t', x')\rangle = R(|t - t'|, |x - x'|)$, where, as in Section 2, the assumptions of statistical homogeneity and stationarity are reflected in the functional dependence only on time and space differences. The specific form of the correlation function $R(t, x)$ to be used here is most conveniently expressed in terms of its spatial Fourier transform

$$R(t, x) = 2\,\bar{V}^2 \int_{\delta}^{\Lambda_o} k^{-\varepsilon} \cos(kx)\ \exp(-ak^{1+\varepsilon}|t|)dk, \tag{24}$$

where a and $\bar{V}$ are constants. The limits Λ_o and δ serve as small- and large-scale cutoffs, respectively, that model in a crude way the fall-off of the energy spectrum at high wavenumbers as a result of dissipation, and at low wavenumbers as a result of the finite size of a real system. The long-time, large-distance solution will be investigated by taking the limit $\delta \to 0$ with Λ_o fixed. The most important property of Equation 24 in that limit is the dominance of high wavenumbers when $\varepsilon < 0$ and of low wavenumbers when $\varepsilon > 0$. It will be seen that this change as ε passes through zero causes a corresponding qualitative change in the large-scale solution for T. The expression in Equation 24 is a slightly simplified version of the Avellaneda & Majda (1990) correlation; this simplification does not include the correlation consistent with the Kolmogorov inertial range and so none of the solutions studied here represent convection of a passive scalar by real turbulence.

The exact solutions of Avellaneda & Majda (1990) were derived using functional-integral techniques; to avoid this complication we consider only the zero-diffusivity case, which may be solved using the method of characteristics (Horntrop & Majda 1994, Elliot et al 1997). [For the most part, the zero-diffusivity solutions are identical to the non-zero-diffusivity solutions derived by Avellaneda & Majda (1990) because random-convection effects generally dominate viscous-diffusion effects.] With the exact solution in hand, moments may be evaluated in the long-time, large-distance limit and compared with the RG solutions.

Consider, then, Equation 23 with $\kappa_o = 0$. Expressing the solution in terms of a Green's function for convenience and choosing the specific initial condition

$G(t = 0, x, y; x', y') = \delta(x - x')\,\delta(y - y')$, the method of characteristics (Garabedian 1964) yields

$$G(t, x, y; x', y') = \delta(x - x')\,\delta\left(y - y' - \int_0^t v(s, x)\,ds\right). \tag{25}$$

Focusing on the moment $\langle T(t, x, y)\rangle = \int dx'dy'\, T_o(x', y')\langle G(t, x, y; x', y')\rangle$, it is necessary to average the Green's function (Equation 25) over the ensemble of velocities $v(t, x)$. Expressing the second δ function in Equation 25 as a Fourier integral according to the relation $\delta(x) = (1/2\pi)\int_{-\infty}^{+\infty} dk\ \exp(ikx)$, the average $\langle G\rangle$ contains the random velocity $v(t, x)$ only in the factor $\langle\exp(-ik\int_0^t v(s, x)\,ds)\rangle$. This factor is straightforwardly evaluated in terms of the correlation function $R(t, x)$ of the velocity field: Approximating the integral by a sum, expanding the exponential in a series and averaging over realizations of $v(t, x)$, one is left with only even products of $v(t, x)$ because of its mean-zero Gaussian statistics. After these manipulations, the averaged series is identical to the Taylor expansion of $\exp[-k^2\int_0^t (t - s)R(s, 0)\,ds]$ (Avellaneda & Majda 1994). The averaged Green's function is thus

$$\langle G(t, x, y; x', y')\rangle = \frac{1}{2\pi}\,\delta(x - x')\int_{-\infty}^{+\infty} dk\exp(ik(y - y')) \times \exp\left(-k^2\int_0^t (t - s)\,R(s, 0)\,ds\right). \tag{26}$$

The long-time, large-distance limit of this solution may now be derived. A precise definition of such a limit was given by Avellaneda & Majda (1990) in terms of a transformation to scaled variables $\tilde{x} = \delta x$, $\tilde{y} = \delta y$, and $\tilde{t} = \rho^2(\delta)t$ (and $\tilde{k} = k/\delta$) that are on the order of one as δ tends to zero. The function $\rho(\delta)$, which must tend to zero with δ, is chosen to make this limit yield finite results.

Introducing these scaled variables and $R(t, x)$ (as given in Equation 24) into Equation 26, one finds

$$-k^2\int_0^t (t - s)R(s, 0)ds = -\frac{\delta^2}{\rho^{6-4/(1+\varepsilon)}}\frac{2\tilde{k}^2\bar{V}^2}{(1 + \varepsilon)a^{-1+2/(1+\varepsilon)}}\tilde{t}^{3-2/(1+\varepsilon)} \times \int_{\alpha_1}^{\alpha_2} y^{-4+2/(1+\varepsilon)}\,F(y)\,dy. \tag{27}$$

Here, we have defined $F(y) = \int_0^y dw\,(y - w)e^{-w}$ and the limits of integration are $\alpha_1 = \rho^{-2}\delta^{1+\varepsilon}a\tilde{t}$ and $\alpha_2 = \rho^{-2}\Lambda_o^{1+\varepsilon}a\tilde{t}$. As $\delta \to 0$, the upper limit of integration tends to infinity and the lower limit may tend to zero, a finite number, or infinity, depending on the behavior of $\rho(\delta)$. The case of the lower limit tending to infinity leads to the simple solutions that are treated here.

When the lower limit tends to infinity, the entire domain of integration is over large values of y, where $F(y) \sim y$. To leading order in δ, the integral becomes

$$-k^2 \int_0^t (t-s)\, R(s,0)\, ds = -\frac{\bar{V}^2}{a} \left(\frac{\delta}{\rho}\right)^2 \left(-\frac{1}{\varepsilon}\right) \left(\Lambda_o^{-2\varepsilon} - \delta^{-2\varepsilon}\right) \tilde{k}^2\, \tilde{t}. \quad (28)$$

The scaling function $\rho(\delta)$ may now be chosen so that this expression is finite as $\delta \to 0$, leading to a nontrivial averaged Green's function. The factor $\Lambda_o^{-2\varepsilon} - \delta^{-2\varepsilon}$ is dominated as δ tends to zero by the first or second term depending on whether ε is less than or greater than zero, respectively. In either case, Equation 28 reduces to $-C_\pm \tilde{k}^2 \tilde{t}$ ($C_\pm$ is a different order-one constant in each case), and the averaged Green's function (Equation 26) reduces to

$$\langle G(\tilde{t}, \tilde{x}, \tilde{y}; \tilde{x}', \tilde{y}') \rangle = \frac{1}{2\pi}\, \delta(\tilde{x} - \tilde{x}') \times \int_{-\infty}^{+\infty} d\tilde{k} \exp(i\tilde{k}(\tilde{y} - \tilde{y}')) \exp\left(-C_\pm \tilde{k}^2 \tilde{t}\right), \quad (29)$$

which is the Green's function for the linear diffusion equation $\partial \bar{T}/\partial \tilde{t} + C_\pm \partial^2 \bar{T}/\partial \tilde{y}^2 = 0$, where $\bar{T} = \lim_{\delta \to 0} \langle T(\tilde{x}, \tilde{y}, \tilde{t}) \rangle$ denotes the large-distance, long-time limit of $\langle T \rangle$.

For $\varepsilon < 0$, the choice $\rho = \delta$ renders Equation 28 finite and $C_- = \bar{V}^2 \Lambda_o^{-2\varepsilon} / (-a\varepsilon)$. The solution is consistent only if the lower limit of the integral in Equation 27 tends to infinity, as was assumed; this additional constraint, that $\delta^{1+\varepsilon}/\rho^2 \to \infty$, is satisfied as long as $\varepsilon < 0$. Avellaneda & Majda (1990) call this Region I in their discussion of the $\kappa_o \neq 0$ problem; their solution differs by the presence of κ_o in C_-.

When $\varepsilon > 0$, the now-dominant factor $\delta^{-2\varepsilon}$ in Equation 28 forces the choice $\rho = \delta^{1-\varepsilon}$ in order that Equation 28 be finite, and one finds $C_+ = \bar{V}^2/(a\varepsilon)$. The lower limit of the integral in Equation 27 tends to infinity, as it must, provided $\varepsilon < 1/3$; the validity of this solution is thus limited to $0 < \varepsilon < 1/3$. This case corresponds to Region II in Avellaneda & Majda's (1990) analysis, and the presence of a nonzero molecular diffusivity does not change this solution because diffusion effects are overwhelmed by convection effects.

The change in scaling from $\rho = \delta$ to $\rho = \delta^{1-\varepsilon}$ as ε passes through zero signifies a fairly dramatic change in the nature of the large-scale solution. The $\rho = \delta$ scaling for $\varepsilon < 0$ implies that a length l (such as the spatial extent of a lump of the scalar T) increases as the square-root of time: $l \propto t^{1/2}$, the standard diffusion relationship. When $\varepsilon > 0$ and the scaling $\rho = \delta^{1-\varepsilon}$ applies, however, length and time scales are related by $l \propto t^{1/(2-2\varepsilon)}$, lengths increase faster with time, and the passive scalar T diffuses more rapidly than it does when $\varepsilon < 0$. This enhanced diffusion reflects the enhanced correlation of the convecting velocity field as $\delta \to 0$ when $\varepsilon > 0$.

Alternative passive-scalar problems have been studied (e.g. Zhang & Glimm 1992) for which time-dependent low-wavenumber cutoffs are chosen so that the large-distance, long-time solution for the mean is, like the two-point correlation, self-similar; this property is not exhibited by the Avellaneda and Majda problem or by real turbulent diffusion. Another alternative problem was suggested by Wallstrom (1995), who argued that the limit $\Lambda_o \to \infty$ with δ fixed is more realistic physically. The issues raised by these authors have no effect on the usefulness of the model problem of Avellaneda and Majda for assessing statistical approximations.

3.2 *Mode-Elimination Renormalization-Group Analysis*

The application of the mode-elimination RG technique to Avellaneda & Majda's passive-scalar problem contains only two basic elements, in contrast to the variety of concepts brought in to carry through the RG analysis of the Navier-Stokes equations. The first element is the mode-elimination scheme itself. The second element is the distant-interaction approximation, which is sufficient to make the RG procedure tractable for this problem. However, the absence of a renormalized force to represent local interactions means that the rationale for the distant-interaction approximation must be different. It will be seen that the RG method works well for some values of the parameter ε but not for others. The following analysis is after Avellaneda & Majda (1992a).

The Fourier transform of Equation 23 for the passive scalar is

$$\hat{T}(\mathbf{k},\omega) = -\frac{ik_2}{(2\pi)^2}\, g_o(\mathbf{k},\omega) \int dp \int d\Omega\, \hat{v}(p,\Omega) \times \hat{T}(k_1 - p, k_2, \omega - \Omega) + g_o(\mathbf{k},\omega)\hat{T}_o(\mathbf{k}), \tag{30}$$

where $g_o(\mathbf{k},\omega) = -1/i\omega$ is the Green's function, k_1 and k_2 denote the components of the wavevector corresponding to x and y, and $k = |\mathbf{k}|$.

The mode-elimination procedure may be set up as in Section 2, with the upper wavenumber cutoff denoted Λ_o. Then modes are eliminated a shell $[\Lambda - \Delta\Lambda, \Lambda]$ at a time from $\Lambda = \Lambda_o$ to $\Lambda = 0$. As modes are eliminated, the renormalized coefficients are gradually incremented; they are accordingly functions of Λ. The only renormalized coefficient to appear in the present analysis is a diffusion coefficient $\kappa(\Lambda)$ measuring diffusion in the y direction that appears in a new term $\kappa(\Lambda)k_2^2\hat{T}$; to streamline the analysis this term is added now to Equation 30 by replacing the bare Green's function by the renormalized Green's function $g(\mathbf{k},\omega) = 1/(-i\omega + k_2^2\kappa(\Lambda))$. The absence of this new term in the original equation implies $\kappa(\Lambda_o) = 0$.

Introducing the notation $\hat{T}^<$, $\hat{T}^>$, and $\hat{v}^<$, $\hat{v}^>$ analogous to Equations 7 and 8 of Section 2, the elimination of modes in the shell may be carried out by constructing an approximate solution for $\hat{T}^>(\mathbf{k},\omega)$ and using it to eliminate $\hat{T}^>$

from the equation for $T^<$. When the equation for $T^<$ is averaged over velocity modes in the shell, all reference to the shell is lost except for an increment to the renormalized diffusivity; summing these increments for all the eliminated shells yields the total diffusivity and a renormalized equation for the large-scale limit of $\hat{T}$. As in the case of the Navier-Stokes equations, the summation of finite increments is replaced by a differential equation for $\kappa(\Lambda)$ when the shell thickness tends to zero.

Use of the distant-interaction approximation at this early stage serves to prevent inclusion of terms that would only turn out to be unimportant (in this approximation) later on. Distant interactions are those whose wavenumber triangles are long and thin, with two long sides and one short side. Accordingly, equations for $\hat{T}^<$ and $\hat{T}^>$ are constructed for which the only interaction terms retained are those whose wavenumber triads contain two (large) wavenumbers in the shell and a third (small) wavenumber outside the shell. Then, introducing the parameter λ_o to play the same role here as it did in Section 2, the equation for $\hat{T}^<(\mathbf{k}, \omega)$ is

$$\hat{T}^<(\mathbf{k}, \omega) = g(\mathbf{k}, \omega)\hat{T}_o^<(\mathbf{k}) - \frac{ik_2}{(2\pi)^2}\lambda_o\, g(\mathbf{k}, \omega) \times \int dp \int d\Omega\, \hat{v}^<(p, \Omega)\hat{T}^<(k_1 - p, k_2, \omega - \Omega) - \frac{ik_2}{(2\pi)^2}\lambda_o\, g(\mathbf{k}, \omega)\int dp \int d\Omega\, \hat{v}^>(p, \Omega)\hat{T}^>(k_1 - p, k_2, \omega - \Omega). \tag{31}$$

The first term on the right-hand side represents all those interactions outside the shell to be dealt with at later steps in the iteration, while the second term represents interactions whose triangles have two long sides (in the shell) and a short side (**k**, out of the shell).

When **k** is one of the long sides of the triangle, the distant-interaction hypothesis removes all interactions except those with a second long side in the shell and a short side out of the shell. The equation for $\hat{T}^>$ thus becomes

$$\hat{T}^>(\mathbf{k}, \omega) = g(\mathbf{k}, \omega)\hat{T}_o^>(\mathbf{k}) - \frac{ik_2}{(2\pi)^2}\lambda_o\, g(\mathbf{k}, \omega) \times \int dp \int d\Omega\, \hat{v}^>(p, \Omega)\hat{T}^<(k_1 - p, k_2, \omega - \Omega) - \frac{ik_2}{(2\pi)^2}\lambda_o\, g(\mathbf{k}, \omega)\int dp \int d\Omega\, \hat{v}(p, \Omega)\hat{T}^>(k_1 - p, k_2, \omega - \Omega). \tag{32}$$

A series expansion for $\hat{T}^>$ carried through terms linear in λ_o leads to an

approximate solution identical to Equation 32 except that the $\hat{T}^>$ in the second term on the right-hand side is replaced by $g\hat{T}_o^>$.

This solution may now be used to eliminate $\hat{T}^>$ from Equation 31. Averaging over $\hat{v}^>$ eliminates the only remaining references to the shell in Equation 31; the equation for $\hat{T}^<$ is then

$$\begin{aligned}\hat{T}^<(\mathbf{k},\omega) = g(\mathbf{k},\omega)\hat{T}_o^<(\mathbf{k}) - \frac{ik_2}{(2\pi)^2}\lambda_o\, g(\mathbf{k},\omega)\int dp \int d\Omega\, \hat{v}(p,\Omega) \\ \times \hat{T}^<(k_1-p,k_2,\omega-\Omega) - \frac{k_2^2}{(2\pi)^4}\lambda_o^2\, g(\mathbf{k},\omega) \\ \times \int dp \int d\Omega \int dp' \int d\Omega' \; < v^>(p,\Omega)\, v^>(p',\Omega') \\ > g(k_1-p,k_2,\omega-\Omega)\hat{T}^<(k_1-p-p',k_2,\omega-\Omega-\Omega'),\end{aligned} \tag{33}$$

an equation defined for $k < \Lambda - \Delta\Lambda$, just as the original Equation 30 was defined for $k < \Lambda$. As for the analysis of the Navier-Stokes equations, the goal is to absorb any new terms into terms already present in the original equation, and so to permit the process of shell elimination to be repeated indefinitely. This requires only that the second term on the right-hand side of Equation 33 be put in a form identical to that of a term already in the original equation. Naturally, that term in the original equation is the $\kappa k_2^2\,\hat{T}$ term inserted precisely for this purpose.

The new term does reduce to the desired form, as a consequence of the homogeneity and stationarity of the statistics of $\hat{v}^>$. The Fourier-velocity correlation may be expressed as $\langle \hat{v}^>(k_1,\omega)\; \hat{v}^>(k_1',\omega')\rangle = (2\pi)^2 E(k_1,\omega)\delta(k_1 + k_1')\;\delta(\omega+\omega')$, where $E(k_1,\omega)$ is the Fourier transform of the correlation function given in Equation 24 restricted to wavenumbers in the shell. The δ functions in this correlation serve to eliminate one pair of integrals in such a way that the term may be written $-k_2^2 g(\mathbf{k},\omega)\;\Delta\kappa\hat{T}$, the desired form. The integral represented by $\Delta\kappa$ is

$$\Delta\kappa = \frac{1}{(2\pi)^2}\int dp \int d\Omega\; E(p,\Omega)\; g(k_1-p,k_2,\omega-\Omega) \tag{34}$$

and the domain of integration is such that both $(k_1 - p, k_2)$ and $(p, 0)$ are in the shell. The distant-interaction approximation may be used to simplify this integral, as was the analogous integral (Equation 14) in Section 2. Here, however, the first term in a formal expansion of the integrand in k/p does not integrate to zero and the approximate evaluation of the integral results from setting $\mathbf{k} = 0$. Once $\mathbf{k}$ is set to zero, the integration domain is $\Lambda - \Delta\Lambda < p < \Lambda$. The frequency ω is also set to zero, because a long-time, as well

as large-distance, approximation is sought. Performing the Ω-integration by contour-integral methods then yields $\Delta\kappa = I(\Lambda - \Delta\Lambda, \Lambda)$, where

$$I(\beta_1, \beta_2) = 2\,\frac{\bar{V}^2}{a}\int_{\beta_1}^{\beta_2} p^{-1-2\varepsilon}dp = \frac{\bar{V}^2}{a}\left(-\frac{1}{\varepsilon}\right)\left(\beta_2^{-2\varepsilon} - \beta_1^{-2\varepsilon}\right). \tag{35}$$

The analysis has culminated in an iterable renormalization-group operation: given the original equation containing the renormalized diffusivity κ and defined for $0 < k < \Lambda$ (Equation 30 with g_o replaced by g), the elimination of the shell $\Lambda - \Delta\Lambda < k < \Lambda$ has the sole effect of replacing κ by $\kappa + \Delta\kappa$ and reducing the range of k to $0 < k < \Lambda - \Delta\Lambda$. With this group operation in hand, shells may be eliminated indefinitely simply by repeatedly renormalizing κ. As in Section 2, the iteration is implemented by letting the shell thickness tend to zero, giving a differential equation for κ as a function of the wavenumber cutoff Λ:

$$\frac{d\kappa}{d\Lambda} \equiv \lim_{\Delta\Lambda\to 0}\frac{\Delta\kappa}{\Delta\Lambda} = -2\,\frac{\bar{V}^2}{a}\Lambda^{-1-2\varepsilon}. \tag{36}$$

Integration yields $\kappa(\Lambda) = I(\Lambda, \Lambda_o)$; the upper limit of integration is Λ_o, to satisfy the boundary condition $\kappa(\Lambda_o) = 0$. It is worth noting that the structure of this simple problem is such that κ does not appear in $\Delta\kappa$ once the distant interaction approximation is invoked. The elimination of each shell is thus independent of all others and one could just as well eliminate one big shell rather than iteratively eliminate a large number of small ones. This was not true of the Navier-Stokes analysis and is not true in general.

To complete the derivation of the large-scale solution, let the elimination of modes be stopped at $\Lambda = \delta$, the parameter used to obtain the large-scale limit in the discussion of the exact solution of Section 3.1. Then the result of the RG analysis is the renormalized equation

$$\frac{\partial\bar{T}}{\partial\tilde{t}} + \frac{\delta^2\kappa(\delta)}{\rho^2(\delta)}\frac{\partial^2\bar{T}}{\partial\tilde{y}^2} = 0, \tag{37}$$

where $\rho(\delta)$, $\bar{T}$ and the tilde-ed variables are as defined in Section 3.1. The behavior of $\kappa(\delta)$ may be studied in the limit $\delta \to 0$ and $\rho(\delta)$ chosen to yield a finite limit.

The renormalized diffusivity $\kappa(\delta) = I(\delta, \Lambda_o)$ takes two forms in the limit $\delta \to 0$, depending on whether ε is positive or negative. If $\varepsilon < 0$, $\kappa(\delta)$ is finite as $\delta \to 0$ and $\kappa(0)$ is identical to the diffusivity found for $\varepsilon < 0$ in the exact theory; the same scaling $\rho(\delta) = \delta$ makes the effective equation (Equation 37) finite in this limit. If $\varepsilon > 0$, $\kappa \sim \delta^{-2\varepsilon}$ becomes infinite as $\delta \to 0$ and the choice $\rho^2(\delta) = \delta^{2-2\varepsilon}$ must be made to yield Equation 37 finite. This scaling and the corresponding diffusion coefficient, $\bar{V}^2/(a\varepsilon)$, are also exact, but the RG solution fails to recognize the boundary at $\varepsilon = 1/3$. In fact, the RG method predicts that the solution for $0 < \varepsilon < 1/3$ is valid for $\varepsilon > 1/3$, where an entirely

different solution arises from the exact formulation given in Section 3.1. The full problem investigated by Avellaneda & Majda contains five qualitatively distinct types of solutions; only the solutions analogous to the two presented here are reproduced by the RG procedure.

3.3 *Discussion*

To clarify those cases where the RG technique does work, a scaling argument similar to that at the end of Section 3.1 may be applied to the velocity correlation (Equation 24) to compare length and time scales with those of the large-scale solution for $\langle T \rangle$. The spatial separation, l_v, and time difference, t_v, of the velocity correlation are found to be related by $t_v \propto l_v^{1+\varepsilon} \propto \delta^{-1-\varepsilon}$. Using the scaling relations at the end of Section 3.1, and letting t_T characterize the time scale of $\langle T \rangle$, one finds $t_v/t_T \propto \delta^{1-\varepsilon}$ when $\varepsilon < 0$ and $t_v/t_T \propto \delta^{1-3\varepsilon}$ when $\varepsilon > 0$. The velocity correlation time is thus small compared with the time scale of $\langle T \rangle$ for $\varepsilon < 1/3$; it is this separation of time scales (and the statistical homogeneity and stationarity of the velocity field) that permits random-convection effects to be modeled by an eddy-diffusion term with a diffusivity independent of space and time. The additional solutions discussed by Avellaneda & Majda (1990) do not have this property and are not represented well by RG solutions. In particular, their solution for convection by a Kolmogorov inertial-range velocity field shares with the present $\varepsilon = 1/3$ solution the property of similar characteristic times of the velocity correlation and $\langle T \rangle$.

In the RG analyses described in Sections 2 and 3 of this article, the renormalized coefficient increases when $\varepsilon > 0$ as shells are eliminated, and so the coefficient is an increasing function of the cutoff Λ. As suggested in Section 1, the original Wilson RG procedure provides for the rescaling of the dependent and independent variables after the elimination of each shell to compensate for the change in Λ and the corresponding increase in the renormalized coefficient. Avellaneda & Majda (1992a) rescaled variables (by introducing the tilde-ed variables), not at each step in the mode-elimination procedure, but at the end, after the expression for the eddy diffusivity had been found. It is only with rescaling that either of these approaches can yield finite renormalized coefficients when the elimination of shells is repeated indefinitely. As seen in Section 2, Yakhot & Orszag (1986) chose not to repeat the shell-elimination indefinitely (making rescaling unnecessary), but simply to stop, either at a fixed value of Λ, to give an expression useful for turbulence modeling (Section 4), or at $\Lambda = k$, to give an inertial-range solution (as in Equation 20).

For $\varepsilon \to 0$, one sees in the RG analyses of Sections 2 and 3 that the effect of the fluctuations is weak and that it is reasonable to attempt a perturbative solution; this is what the ε-expansion RG is intended to do. The perturbative solution in the Navier-Stokes analysis was found to be justified for small ε; a similar argument applies to the passive scalar example, where, for

the two solutions derived here, the first term of the ε expansion is the exact solution.

4. TURBULENCE MODELING

4.1 *Kolmogorov's Constant*

As noted in Section 1, Fournier & Frisch (1983) and Yakhot & Orszag (1986) extended the RG method to compute turbulence coefficients. The values of the coefficients in the Yakhot-Orszag analysis follow from the ε expansion (Section 2.6) and the steady-state relation between the input parameter D_o and the energy dissipation rate ε.

Using the effective equation (Equation 19), one may derive the dynamical equation for the energy spectrum $E(k,t)$ as a series expansion in powers of λ_o (Dannevik et al 1987). For nondimensional variables, the expansion is in powers of $\bar{\lambda} \propto \varepsilon^{1/2}$ (Equation 17). At second order in the ε expansion, the eddy-damped quasi-normal Markovian (EDQNM) equations (Orszag 1970, Lesieur 1990) are recovered with RG inertial-range $\nu(k)$ given by Equation 20. By equating transport power to energy dissipation rate $\mathcal{E}$ in the statistically steady state, the EDQNM equations with RG coefficients yield $1.59\mathcal{E} \approx 2D_oS_d/(2\pi)^d$ (Dannevik et al 1987, Kraichnan 1971). Substituting this relation into $\nu(k)$ leads to $\nu(k) \sim 0.49\mathcal{E}^{1/3}k^{-4/3}$ with $d = 3$ and $\varepsilon = 4$. The steady-state energy balance at zeroth order in ε then gives $E(k) = 1.61\mathcal{E}^{2/3}k^{-5/3}$ with Kolmogorov constant $C_K = 1.61$ (Yakhot & Orszag 1986), in good agreement with experiments (Sreenivasan 1995).

4.2 *The Renormalization-Group Model for Large Eddy Simulations*

The eddy viscosity $\nu(\Lambda)$ given by Equation 16 may be used for turbulence modeling of real (inhomogeneous) flows. Strictly speaking, Equation 16 is valid only for $\Lambda \ll \Lambda_o$, where it is independent of Λ_o and ν_o. However, Yakhot & Orszag (1986) suggest it as a formula to interpolate between $\nu = \nu_o$ and the inertial range expression $\nu = 0.49\mathcal{E}^{1/3}\Lambda^{-4/3}$. Yakhot et al (1989) converted Equation 16 to an expression suitable for large eddy simulations (LES) (see also Orszag et al 1993). This is achieved by using $1.59\mathcal{E} = 2D_oS_d/(2\pi)^d$ together with the definition of the filter width $\Delta = 2\pi/\Lambda$ and the relation for the Kolmogorov cutoff wavenumber $\Lambda_o = \gamma\mathcal{E}^{1/4}\nu_o^{-3/4}$, where $\gamma = 0.2$ is determined from experimental data. Finally, the approximate relation $\mathcal{E} \approx \nu(\Lambda)(\partial u_i^</\partial x_j + \partial u_j^</\partial x_i)^2/2$ leads to

$$\nu(\Delta) = \nu_o\left[1 + H\left(\frac{0.12\Delta^4}{2\nu_o^3(2\pi)^4}\nu(\Delta)\left(\frac{\partial u_i^<}{\partial x_j} + \frac{\partial u_j^<}{\partial x_i}\right)^2 - C\right)\right]^{1/3}, \qquad (38)$$

where $C = 3(2\pi)^d A_d^0 \mathcal{E}/(16 D_o S_d \gamma^4) \approx 73.5$ and $H(x)$ is the Heaviside function. The Heaviside function gives a jump discontinuity in $\nu(\Delta)$, arising because the RG method assumes that the energy spectrum changes abruptly from $E(k) = 0$ for $k > \Lambda_o$ to $E(k) \propto k^{-5/3}$ for $k < \Lambda_o$. Thus the RG LES model reduces to the molecular viscosity ν_o in regions of low Reynolds number when the argument of the Heaviside function is negative, and turns on abruptly in regions of higher Reynolds number flow. For high Reynolds number, when the filter width is much larger than the Kolmogorov scale, Equation 38 becomes the Smagorinsky (1963) formula $\nu(\Delta) = c_s \Delta^2 S$, where $S = (S_{ij} S_{ij})^{1/2}$, $S_{ij} = (\partial u_i^< / \partial x_j + \partial u_j^< / \partial x_i)/2$, and $c_s \approx 0.003$. A more convenient form of Equation 38 for numerical implementation has been explored by Lund (1990), Piomelli et al (1990b), and Orszag et al (1993).

The ability of the model to turn off in low Reynolds number flow regions suggests that it can be used for laminar-turbulence transition without the need for ad-hoc intermittency or near-wall corrections (e.g. Piomelli et al 1990a). The RG LES model has been used with mixed success to describe transitional and turbulent channel flow (Yakhot et al 1989, Zang & Piomelli 1993), transitional and turbulent flow over a backward facing step (Karniadakis et al 1993), and transitional boundary-layer flow (Piomelli et al 1990a). Because the model was derived under the assumption of isotropy, an ad hoc anisotropy correction was introduced by Yakhot et al (1989) to better accommodate wall flow. The other studies used the RG model (Equation 38) without ad hoc adjustments. As mentioned in Section 2.3, the RG method scale-elimination leads to a stochastic backscatter force with spectrum $D_b(k) \propto k^2$, however, to our knowledge, the RG backscatter force has not been used in LES calculations.

4.3 *The Renormalization Group Model $\mathcal{K}-\mathcal{E}$ Transport Equations*

The Yakhot-Orszag method has also been used to derive $\mathcal{K} - \mathcal{E}$ transport equations for turbulence modeling (Yakhot & Orszag 1986, Smith & Reynolds 1992, Yakhot & Smith 1992). In general, the $\mathcal{K} - \mathcal{E}$ model consists of coupled transport equations for the ensemble-averaged velocity $\langle \mathbf{u} \rangle$, kinetic energy $\mathcal{K}$, and dissipation rate of energy $\mathcal{E}$. The exact equations for $\langle \mathbf{u} \rangle$, $\mathcal{K}$, and $\mathcal{E}$ are not closed, and the standard closure based on dimensional analysis involves an eddy viscosity as well as model source terms in the $\mathcal{K}$ and $\mathcal{E}$ equations, most notably, model $\mathcal{E}$-production and $\mathcal{E}$-destruction terms with empirically determined coefficients (Speziale 1991). For the development of the model $\mathcal{E}$ equation, exact source terms that are estimated to scale as $R_\tau^{1/2}$ (Tennekes & Lumley 1972) are assumed to cancel at leading order such that the model source terms are of order one, where the turbulence Reynolds number R_τ is defined as $R_\tau = \mathcal{K}^2/(\mathcal{E}\nu_o)$.

Yakhot & Orszag (1986) and Yakhot & Smith (1992) used the RG scale-removal procedure and the ε expansion to systematically arrive at model $\mathcal{K}-\mathcal{E}$ equations. The RG $\mathcal{K}-\mathcal{E}$ equations involve the RG eddy viscosity rewritten in terms of $\mathcal{K}$ and $\mathcal{E}$ (Yakhot & Orszag 1986), and model $\mathcal{E}$-production and $\mathcal{E}$-destruction terms with nonadjustable coefficients (Yakhot & Smith 1992). The RG method predicts the $O(R_\tau^{1/2})$ scaling of source terms in the $\mathcal{E}$ equation, as well as their exact cancellation at leading order; the method leads to $\mathcal{E}$-production and $\mathcal{E}$-destruction terms of the same form as the standard model, with different values for the nondimensional coefficients (Smith & Reynolds 1992, Yakhot & Smith 1992). Furthermore, an additional $\mathcal{E}$-production term is predicted (Yakhot & Smith 1992), which is on the same order of magnitude as the standard $\mathcal{E}$-production term in flow regions of large strain rate. Yakhot et al (1992) gave an approximate formula for the new $\mathcal{E}$-production term involving the nondimensional rate of strain $\eta = S\mathcal{K}/\mathcal{E}$ and one adjustable parameter. Thus the RG method provides a theoretical basis for the standard model, and moreover, indicates how the standard model may be improved.

The new $\mathcal{E}$-production term and the modified values of model coefficients derived using the RG method have indeed led to improved performance of the $\mathcal{K}-\mathcal{E}$ model for a variety of flows. Speziale et al (1991) showed that the RG $\mathcal{K}-\mathcal{E}$ model gives better predictions for the growth of $\mathcal{K}$ in homogeneous shear flow (see also Smith & Yakhot 1993). Superior performance for separated flows and flows involving vortex shedding (Choudhury et al 1993) further indicates that the RG model is a step beyond the standard model. As another example, Han & Reitz (1995) applied the RG $\mathcal{K}-\mathcal{E}$ model to variable-density engine flows and found that the presence of the new $\mathcal{E}$-production term allowed for proper prediction of flow structures during spray combustion. Thus, the RG model renders feasible $\mathcal{K}-\mathcal{E}$ modeling of a broader range of practical flow problems than was previously possible.

5. SUMMARY

We have tried to explain the concepts behind the RG scale-removal procedure for the derivation of large-scale, long-time equations of motion for problems related to Navier-Stokes turbulence. Section 2 reviewed in detail the RG iterative removal of scales from the incompressible Navier-Stokes equations in Fourier space, with stochastic forcing characterized by a parameter ε. The specific form of the forcing spectrum introduced by Fournier & Frisch (1983) was further exploited by Yakhot & Orszag (1986) in what has become known as the Yakhot-Orszag RG method. Here we have presented a sensible interpretation of the force as a model for a particular class of nonlinear interactions. We have also discussed the distant-interaction approximation at length, and in this

particular case, its relation to the force. In addition to motivating these two main ingredients of the Yakhot-Orszag method, we have described the possible sources of error involved in the procedure and assessed the consequences of retaining the cubic nonlinearity. Finally, we have discussed the relationship between the RG and the DIA closures.

Even though they may be motivated physically or otherwise, it is evident that many of the steps in the RG scale-removal procedure as currently formulated are mathematically not rigorously justified. As discussed in Section 2.8, a few attempts have been made to understand more deeply the nature of these various approximations, such as the distant-interaction approximation and the ε-expansion. Avellaneda & Majda (1992a) undertook a comprehensive and illuminating study of a model problem of passive scalar convection; their work was reviewed in Section 3. In this case, the distant-interaction approximation together with iterative removal of scales can lead to exact large-scale, long-time results.

Despite the lack of rigor involved in the Yakhot-Orszag procedure, it yields good predictions for turbulence coefficients, such as the Kolmogorov constant, and for turbulence models, such as the model $\mathcal{K} - \mathcal{E}$ transport equations (Section 4). Although this agreement may be fortuitous, it suggests that the method warrants further investigation and development. As it stands, the procedure appears to be viable for the derivation of models for more complicated physics, such as compressible turbulence (Staroselsky et al 1990) and aspects of turbulent combustion (Yakhot 1988).

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